1. In the Arrow-Debreu model, let $u : \mathbb{R}^n_+ \rightarrow \mathbb{R}_+$ be the utility function of a trader with initial endowment $e \in \mathbb{R}^n_+$. Assume $u(\cdot)$ is continuous and strictly concave. Let $p = (p_1, \ldots, p_n) \in \Delta_n$ be the fixed prices. Prove the following properties:

a) Show that this trader has at least one restrictively optimal bundle $x \in [0, 1.1]^n$ with respect to $p$, and it is unique.

b) Show that if $x \in [0, 1.1]^n$ is the restrictively optimal bundle with respect to $p$ and satisfies

$$x_j < 1.1, \quad \text{for all } j \in [n],$$

then $x$ is also a (globally) optimal bundle for this trader.

2. In the Arrow-Debreu model, let $p = (p_1, \ldots, p_n) \in \Delta_n$ be a price vector with $p_j > 0$ for all $j \in [n]$; and let $x_1, \ldots, x_k \in \mathbb{R}^n_+$ be $k$ bundles of goods such that $x_i$ is optimal for trader $i, i \in [k]$. Also assume that the total supply of each good in the market is exactly 1 and all the utility functions are continuous, strictly concave and non-satiated. Show that if

$$\sum_{i \in [k]} x_{i,j} \leq 1, \quad \text{for all } j \in [n],$$

then we must have $\sum_{i \in [k]} x_{i,j} = 1$ for all good $j \in [n]$. (Hint: Use the last exercise from Set 6/7.)

3. (Optional) Given a linear Fisher market in which all parameters (budgets $b_i$ and slopes $u_{i,j}, i \in [k]$ and $j \in [n]$) are rational numbers, show that if there exists a market equilibrium, then there exists a rational market equilibrium. (Hint: Guess the support of $x_i$ for each buyer $i \in [k]$, and then use it to formulate a linear program on $x_{i,j}$ and $p_j, i \in [k]$ and $j \in [n]$. It is worth pointing out that the same idea also works for Fisher markets with piecewise-linear utility functions:

$$u_i(x) = \sum_{j \in [n]} u_{i,j}(x_j),$$

where each $u_{i,j} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is now a piecewise-linear and concave function.)