1. In the Arrow-Debreu model, let $u : \mathbb{R}_+^n \to \mathbb{R}_+$ be the utility function of a trader with initial endowment $\mathbf{e} \in \mathbb{R}_+^n$. Assume $u(\cdot)$ is continuous and strictly concave. Let $\mathbf{p} = (p_1, \ldots, p_n) \in \Delta_n$ be the fixed prices. Prove the following properties:

a) Show that this trader has at least one restrictively optimal bundle $\mathbf{x} \in [0, 1.1]^n$ with respect to $\mathbf{p}$, and it is *unique*.

b) Show that if $\mathbf{x} \in [0, 1.1]^n$ is the restrictively optimal bundle with respect to $\mathbf{p}$ and satisfies

$$x_j < 1.1, \quad \text{for all } j \in [n],$$

then $\mathbf{x}$ is also a (globally) optimal bundle for this trader.

2. In the Arrow-Debreu model, let $\mathbf{p} = (p_1, \ldots, p_n) \in \Delta_n$ be a price vector with $p_j > 0$ for all $j \in [n]$; and let $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{R}_+^n$ be k bundles of goods such that $\mathbf{x}_i$ is optimal for trader i, $i \in [k]$. Also assume that the total supply of each good in the market is exactly 1 and all the utility functions are continuous, strictly concave and non-satiated. Show that if

$$\sum_{i \in [k]} x_{i,j} \le 1, \quad \text{for all good } j \in [n],$$

then we must have $\sum_{i \in [k]} x_{i,j} = 1$ for all good $j \in [n]$. (Hint: Use the last exercise from Set 6/7.)

3. (Optional) Given a linear Fisher market in which all parameters (budgets b_i and slopes $u_{i,j}$, $i \in [k]$ and $j \in [n]$) are rational numbers, show that if there exists a market equilibrium, then there exists a rational market equilibrium. (Hint: Guess the support of $\mathbf{x}_i$ for each buyer $i \in [k]$, and then use it to formulate a linear program on $x_{i,j}$ and p_j, $i \in [k]$ and $j \in [n]$. It is worth pointing out that the same idea also works for Fisher markets with piecewise-linear utility functions:

$$u_i(\mathbf{x}) = \sum_{j \in [n]} u_{i,j}(x_j),$$

where each $u_{i,j} : \mathbb{R}_+ \to \mathbb{R}_+$ is now a piecewise-linear and concave function.)