

1. Let  $A$  be the set of candidates with  $|A| \geq 3$ . Given an incentive compatible social choice function

$$f : L^n \rightarrow A$$

onto  $A$ , where  $L$  is the set of all preferences over the candidates  $A$ , we showed in class a way to construct a social welfare function

$$F : L^n \rightarrow L.$$

Show that if  $f$  is not a dictatorship function, then neither is  $F$ .

2. We have  $2n$  players who live on the same street where a new hospital will be built. Each player has a secret preferred location for the hospital, denoted by  $x_i \in [0, 1]$  for player  $i$ ,  $i \in [2n]$ . If the hospital is built at  $y \in [0, 1]$ , then the cost of player  $i$  is

$$|x_i - y|.$$

Every player wants to minimize his or her cost. The goal is to design an *incentive compatible* mechanism *without money transfer* that can minimize the total cost

$$\sum_{i \in [2n]} |x_i - y|.$$

Consider the following natural mechanism. First we ask the players to report their preferred locations  $x_i$ ,  $i \in [2n]$ . We order these numbers from least to greatest, and let  $\ell$  denote the  $n$ th smallest and  $r$  denote the  $(n + 1)$ th smallest in the  $x_i$ 's reported. Then the hospital is built at  $y = (\ell + r)/2$ .

It is easy to show that if all the players report truthfully their preferred locations, then the output minimizes the total cost. But is this mechanism incentive compatible? If not, give a situation where one of the players can benefit by cheating. Can you modify this mechanism and make it incentive compatible (and still minimize the total cost without money transfer)?