1. Let \( c : 2^A \rightarrow \mathbb{R}_+ \) be a cost function and \( \xi : A \times 2^A \rightarrow \mathbb{R}_+ \) be a cross-monotone cost-sharing scheme, where \( A = \{1, 2, \ldots, n\} \) is the set of players. We showed in class that, given any vector \( (v_1, \ldots, v_n) \in \mathbb{R}_+^n \) of values, there is a unique maximal happy set \( S \subseteq A \) that satisfies
\[
v_i \geq \xi(i, S), \quad \text{for all } i \in S.
\]
Prove that the following simple algorithm always returns the maximal happy set (which is far more efficient than enumerating all possible subsets \( S \) of \( A \)):

(a) Initialize \( S \leftarrow A \)

(b) Repeat

Set \( S \leftarrow \{i \in S : v_i \geq \xi(i, S)\} \)

(or equivalently, remove all the \( i \)'s from \( S \) that satisfy \( v_i < \xi(i, S)\))

(c) Until \( v_i \geq \xi(i, S) \) for all \( i \in S \) or \( S = \emptyset \), and return \( S \)

2. Let \( c \) be a cost function and \( \xi \) be a cross-monotone cost-sharing scheme. We used \( \xi \) to construct a mechanism \( M_\xi \) in class. To prove that it is group incentive compatible, we showed in class that any subset of players cannot manipulate \( M_\xi \) by bidding lower than their true values. Now prove the following lemma which shows that overbidding does not help. (Hint: Start with the case when \( |C| = 1 \).) By combining these two pieces together, we get that \( M_\xi \) is group incentive compatible.

Lemma 1. Let \( \mathbf{v} = (v_1, \ldots, v_n) \) be the true values of the players \( A = \{1, \ldots, n\} \). Let \( C \subseteq A \) be a subset of players and \( \mathbf{v'} = (v'_1, \ldots, v'_n) \) be a new vector in which

1. \( v'_i = v_i \) for all \( i \notin C \); and

2. \( v'_i \geq v_i \) for all \( i \in C \).

(In other words, players in \( C \) overbid.) We let \( u_i \) denote the utility of player \( i \), for every \( i \in A \), when the mechanism \( M_\xi \) sees \( \mathbf{v} \); and \( u'_i \) denote the utility of player \( i \) when \( M_\xi \) sees \( \mathbf{v'} \). Show that if \( u'_i \geq u_i \) for all \( i \in C \), then the maximal happy sets in the two cases are the same (and thus, changing \( \mathbf{v} \) to \( \mathbf{v'} \) does not affect the outcome of \( M_\xi \) and \( u'_i = u_i \) for all \( i \in A \)).