1. Let $c : 2^A \to \mathbb{R}_+$ be a cost function and $\xi : A \times 2^A \to \mathbb{R}_+$ be a cross-monotone cost-sharing scheme, where $A = \{1, 2, \dots, n\}$ is the set of players. We showed in class that, given any vector $(v_1, \dots, v_n) \in \mathbb{R}_+^n$ of values, there is a unique maximal happy set $S \subseteq A$ that satisfies

$$v_i \geq \xi(i, S), \quad \text{for all } i \in S.$$

Prove that the following simple algorithm always returns the maximal happy set (which is far more efficient than enumerating all possible subsets S of A):

(a) Initialize $S \leftarrow A$

(b) Repeat

Set $S \leftarrow \{i \in S : v_i \geq \xi(i, S)\}$
(or equivalently, remove all the i's from S that satisfy $v_i < \xi(i, S)$)

(c) Until $v_i \geq \xi(i, S)$ for all $i \in S$ or $S = \emptyset$, and return S

2. Let c be a cost function and ξ be a cross-monotone cost-sharing scheme. We used ξ to construct a mechanism M_ξ in class. To prove that it is group incentive compatible, we showed in class that any subset of players cannot manipulate M_ξ by biding lower than their true values. Now prove the following lemma which shows that overbidding does not help. (Hint: Start with the case when $|C| = 1$.) By combining these two pieces together, we get that M_ξ is group incentive compatible.

Lemma 1. *Let $\mathbf{v} = (v_1, \dots, v_n)$ be the true values of the players $A = \{1, \dots, n\}$. Let $C \subseteq A$ be a subset of players and $\mathbf{v}' = (v'_1, \dots, v'_n)$ be a new vector in which*

1. *$v'_i = v_i$ for all $i \notin C$; and*
2. *$v'_i \geq v_i$ for all $i \in C$.*

(In other words, players in C overbid.) We let u_i denote the utility of player i, for every $i \in A$, when the mechanism M_ξ sees $\mathbf{v}$; and u'_i denote the utility of player i when M_ξ sees $\mathbf{v}'$. Show that if $u'_i \geq u_i$ for all $i \in C$, then the maximal happy sets in the two cases are the same (and thus, changing $\mathbf{v}$ to $\mathbf{v}'$ does not affect the outcome of M_ξ and $u'_i = u_i$ for all $i \in A$).