1. [Substraction] Let $G' = (A', B')$ be the $2K \times 2K$ generalized matching pennies game, where

$$A'_{2i-1,2i-1} = A'_{2i-1,2i} = A'_{2i,2i-1} = A'_{2i,2i} = M = 2^{K+1}, \quad \text{for all } i \in [K],$$

and $B' = -A'$. Let $G = (A, B)$ be the following $2K \times 2K$ game:

a) $A_{5,5} = A'_{5,5} + 1$, $A_{6,6} = A'_{6,6} + 1$, and $A_{i,j} = A'_{i,j}$ for all other entries of $A$;

b) $B_{1,5} = B'_{1,5} + 1$, $B_{3,6} = B'_{3,6} + 1$, $B_{5,6} = B'_{5,6} + 1$, and $B_{i,j} = B'_{i,j}$ for all other entries of $B$.

Show that every Nash equilibrium $(x, y)$ of $G$ satisfies

$$\min(x_1 - x_3, 1/K) - \epsilon \leq x_5 \leq \max(x_1 - x_3, 0) + \epsilon, \quad \text{where } \epsilon = 1/2^K.$$

2. [Boolean Or] Let $G' = (A', B')$ be the $2K \times 2K$ generalized matching pennies game defined above. Let $G = (A, B)$ be the following $2K \times 2K$ game:

a) $A_{5,5} = A'_{5,5} + 1$, $A_{6,6} = A'_{6,6} + 1$, and $A_{i,j} = A'_{i,j}$ for all other entries of $A$;

b) $B_{1,5} = B'_{1,5} + 1$, $B_{3,6} = B'_{3,6} + 1$, $B_{5,6} = B'_{5,6} + 1/(2K)$ for all $\ell \in [2K],
\quad \text{and } B_{i,j} = B'_{i,j}$ for all other entries of $B$.

Show that every Nash equilibrium $(x, y)$ of $G$ satisfies

1. If $1/K - \epsilon \leq x_1 \leq 1/K + \epsilon$ or $1/K - \epsilon \leq x_3 \leq 1/K + \epsilon$, then $1/K - \epsilon \leq x_5 \leq 1/K + \epsilon$; and

2. If $x_1 \leq \epsilon$ and $x_3 \leq \epsilon$, then $x_5 = 0$,

where $\epsilon = 1/2^K$.

3. Given any Arrow-Debreu market, show that if $p = (p_1, \ldots, p_n)$ is a market equilibrium with $\sum_{i \in [n]} p_i > 0$, then one can normalize it to be $p' = (p'_1, \ldots, p'_n)$:

$$p'_i = \frac{p_i}{\sum_{i \in [n]} p_i}, \quad \text{for all } i \in [n],$$

and $p'$ is also a market equilibrium. (Thus, when computing an equilibrium, we can restrict the search space to be $\Delta_n$.)

4. Let $p = (p_1, \ldots, p_n)$ be a price vector and $u(\cdot) : \mathbb{R}^n_+ \to \mathbb{R}_+$ be the utility function of a trader with budget $b > 0$. If

1. $u(\cdot)$ is both strictly concave and non-satiated; and

2. $u(\cdot)$ has an optimal bundle $x \in \mathbb{R}^n_+$ with respect to $p$ and $b$,

then we must have $p \cdot x = b$. 

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