1. [Substraction] Let $G' = (\mathbf{A}', \mathbf{B}')$ be the $2K \times 2K$ generalized matching pennies game, where

$$A'_{2i-1,2i-1} = A'_{2i-1,2i} = A'_{2i,2i-1} = A'_{2i,2i} = M = 2^{K+1}, \quad \text{for all } i \in [K],$$

and $\mathbf{B}' = -\mathbf{A}'$. Let $G = (\mathbf{A}, \mathbf{B})$ be the following $2K \times 2K$ game:

a) $A_{5,5} = A'_{5,5} + 1$, $A_{6,6} = A'_{6,6} + 1$, and $A_{i,j} = A'_{i,j}$ for all other entries of $\mathbf{A}$;

b) $B_{1,5} = B'_{1,5} + 1$, $B_{3,6} = B'_{3,6} + 1$, $B_{5,6} = B'_{5,6} + 1$, and $B_{i,j} = B'_{i,j}$ for all other entries of $\mathbf{B}$.

Show that every Nash equilibrium $(\mathbf{x}, \mathbf{y})$ of G satisfies

$$\min(x_1 - x_3, 1/K) - \epsilon \le x_5 \le \max(x_1 - x_3, 0) + \epsilon, \quad \text{where } \epsilon = 1/2^K.$$

2. [Boolean Or] Let $G' = (\mathbf{A}', \mathbf{B}')$ be the $2K \times 2K$ generalized matching pennies game defined above. Let $G = (\mathbf{A}, \mathbf{B})$ be the following $2K \times 2K$ game:

a) $A_{5,5} = A'_{5,5} + 1$, $A_{6,6} = A'_{6,6} + 1$, and $A_{i,j} = A'_{i,j}$ for all other entries of $\mathbf{A}$;

b) $B_{1,5} = B'_{1,5} + 1$, $B_{3,5} = B'_{3,5} + 1$, $B_{\ell,6} = B'_{\ell,6} + 1/(2K)$ for all $\ell \in [2K]$, and $B_{i,j} = B'_{i,j}$ for all other entries of $\mathbf{B}$.

Show that every Nash equilibrium $(\mathbf{x}, \mathbf{y})$ of G satisfies

1. If $1/K - \epsilon \le x_1 \le 1/K + \epsilon$ or $1/K - \epsilon \le x_3 \le 1/K + \epsilon$, then $1/K - \epsilon \le x_5 \le 1/K + \epsilon$; and
2. If $x_1 \le \epsilon$ and $x_3 \le \epsilon$, then $x_5 = 0$,

where $\epsilon = 1/2^K$.

3. Given any Arrow-Debreu market, show that if $\mathbf{p} = (p_1, \dots, p_n)$ is a market equilibrium with $\sum_{i \in [n]} p_i > 0$, then one can normalize it to be $\mathbf{p}' = (p'_1, \dots, p'_n)$:

$$p'_i = \frac{p_i}{\sum_{i \in [n]} p_i}, \quad \text{for all } i \in [n],$$

and $\mathbf{p}'$ is also a market equilibrium. (Thus, when computing an equilibrium, we can restrict the search space to be Δ_n.)

4. Let $\mathbf{p} = (p_1, \dots, p_n)$ be a price vector and $u(\cdot) : \mathbb{R}^n_+ \to \mathbb{R}_+$ be the utility function of a trader with budget $b > 0$. If

1. $u(\cdot)$ is both strictly concave and non-satiated; and
2. $u(\cdot)$ has an optimal bundle $\mathbf{x} \in \mathbb{R}^n_+$ with respect to $\mathbf{p}$ and b,

then we must have $\mathbf{p} \cdot \mathbf{x} = b$.