COMS E6998: Algorithmic Game Theory and Economics — Jan 20, 2011

Lecture 1 – Introduction

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1 Subject Matter of the Course

Game theory is the theory of competitions. It is based on the premise that the competitors act selfishly and rationally. In many applications, it is important to understand or even predict the behavior of selfish players and the outcomes of competitions.

Game theory is a mathematical field and particularly used in economics, science, and engineering. Computer scientists became interested in game theory with the advent of the internet and electronic commerce. Before these technological developments there was no need to apply game-theoretical ideas to computer science because computers were not connected and thus there was no competition.

2 Three Parts of the Course

This course will focus on three major parts of game-theoretic reasoning:

1. Dynamics — States evolve over time and changing strategies may affect the state.
2. Computation — The computation of solution concepts.
3. Design — Game theory is not only about analyzing systems, but also about designing systems.

3 Normal Form Games

If we talk about games in the sense of game theory, we mean normal form games defined as follows.

Definition 1. *Normal form games are characterized by the following elements:*

1. *There is a finite set of k players $K = \{1, 2, \ldots, k\}$.*
2. *Each of the k players has a set of finite strategies $S_i = \{1, 2, \ldots, n_i\}$, $i \in [k]$. For example, in the game rock-paper-scissors, every player has three strategies. We will use $s_i \in S_i$ to denote a strategy of player i, and use $S = S_1 \times S_2 \times \cdots \times S_k$ to denote the strategy space of the game.*
3. *Each player has a utility function u_i, $i \in [k]$. Once all players made their decisions $\mathbf{s} = (s_1, \ldots, s_k) \in S$, the utility of player i is given by $u_i(\mathbf{s})$. Every player wants to maximize his or her own utility. Instead of utility functions, some games use cost functions, which every player wants to minimize.*

4 Split-or-Steal Game

Game theory aims to model situations in which players interact or affect each other's outcome. For example, in the split-or-steal game two players compete for a money prize whose amount does not only depend on their own strategy, but also on the strategy the other player selects. If both players decide to steal the money, they will not get anything. If both players decide to split the money, they will each get \$50k. However, if one of them decides to steal while the other decides to split, the player who steals will receive \$100k and the player who splits will lose \$10k. (see http://www.youtube.com/watch?v=p3Uos2fzIJ0)

The outcome of the game can be represented in the following payoff matrix:

	Steal	**Split**
Steal	0 / 0	\$100k / - \$10k
Split	- \$10k / \$100k	\$50k / \$50k

Figure 1: The Split-or-Steal Game

5 Basic Solution Concepts

Two basic solution concepts for normal form games are dominant strategy solutions and pure equilibria. Both concepts will be discussed in turn.

5.1 Dominant Strategy Solutions

From the payoff matrix in Figure 1, it can be observed that the best strategy for both players is to steal. Regardless of what the opponent chooses, each player always receives a higher payoff by stealing; that is to say that stealing is the dominant strategy. The split-or-steal game as well as the prisoner's dilemma have the special property that each player has a dominant strategy. We say that a game has a dominant strategy solution if it has this property.

To define dominant strategy solutions formally, we need the following notation. Given any player $i \in [k]$, we use $\mathbf{s}_{-i}$ to denote a vector in $S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_k$, in which every player picks a strategy except player i. Let $s_i \in S_i$ be a strategy of player i. We will use $(s_i, \mathbf{s}_{-i}) \in S$ to denote the strategy vector combined from s_i and $\mathbf{s}_{-i}$ and thus, $u_i(s_i, \mathbf{s}_{-i})$ is well defined.

Definition 2. *A strategy vector $\mathbf{s}^* = (s_1^*, \ldots, s_k^*) \in S$ is a dominant strategy solution, if for any player i, and any vector $\mathbf{s}_{-i} \in S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_k$, we have:*

$$u_i(s_i^*, \mathbf{s}_{-i}) \geq u_i(s_i, \mathbf{s}_{-i}), \qquad \textit{for any } s_i \in S_i.$$

5.2 Pure Equilibria

A weaker strategy concept — compared to dominant strategy solutions — are pure equilibria. A pure equilibrium means that no player has anything to gain by changing only his or her own strategy unilaterally. If each player has chosen a strategy and no player can benefit by changing only his or her strategy, then the current set of strategy choices constitutes a pure equilibrium. Therefore, it follows that every dominant strategy solution is also a pure equilibrium. However, the reverse is not true. Not every dominant strategy is also a pure equilibrium.

Definition 3. *A strategy vector* $\mathbf{s}^* = (s_1^*, \ldots, s_k^*) \in S$ *is a pure equilibrium, if for any player* i*, we have*

$$u_i(\mathbf{s}^*) \geq u_i(s_i, \mathbf{s}_{-i}^*), \qquad \textit{for any } s_i \in S_i,$$

where $\mathbf{s}_{-i}^*$ *denotes the* $(k-1)$*-dimensional vector obtained from* $\mathbf{s}^*$ *after removing* s_i^**.*

5.3 Advantages and Disadvantages of Pure Equilibria

A pure equilibrium is one of the solution concepts (which aim to describe mathematically what outcomes are more likely to happen) that we will see in the course. However, for general normal form games, it has the following advantages and disadvantages.

Advantages	**Disadvantages**
are stable	do not always exist
	can be very inefficient (e.g., consider the split-or-steal game)
	can have multiple pure equilibria

Figure 2: Advantages and Disadvantages of Pure Equilibria

Next we will focus on a special subclass of normal form games called global network connection games. We will show that for these games pure equilibria make a lot more sense.

6 Global-Network-Connection Games

In a global-network-connection game, we are given a graph $G = (V, E)$ and k players. Each player i has the goal to build a path from the source node s_i to the sink node t_i. Players can have the same/different source or sink node. For example, below player 1 has the goal to build a path from s_1 to t_1, while player 2 needs to build a path from s_2 to t_2.

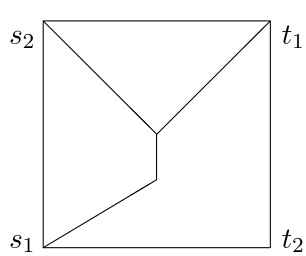

Figure 3: $G = (V, E)$

For each player i, he or she can pick any path P_i connecting s_i and t_i. So his or her strategy set is the set of paths P_i connecting s_i and t_i. In the global-network-connection game, instead of maximizing utility, each player wants to minimize his or her cost. The costs for building the paths are allocated such that the edges that are used exclusively by one player must be paid in full by that player. However, for the edges that are used by multiple players, each player is required to contribute only partially to the costs.

Definition 4. *Let $(P_1, \ldots, P_k)$ denote a strategy vector where P_i is a path connecting s_i and t_i, where $i \in [k]$. For each edge $e \in E$, we let k_e denote the number of players using edge $e \in E$ in their paths and $c_e \geq 0$ denote the cost of using edge e, then we assign a cost share of $\frac{c_e}{k_e}$ to each player who uses e. Thus, the total cost incurred by player i under $(P_1, \ldots, P_k)$ are given by:*

$$cost_i(P_1, \ldots, P_k) = \sum_{e \in P_i} \frac{c_e}{k_e}.$$

7 Potential Functions

The global-network-connection game is a potential game, meaning that the incentive of all players to change their strategy can be expressed in one global function, which is called the potential function. This function has the characteristic that if a player changes his or her strategy, the change in the potential function is simultaneously reflected in the change of cost of that player.

Definition 5. *We can define a potential function f as follows: Suppose we have a game that satisfies*

$$f(P_1, \ldots, P_k) - f(P_1, \ldots, P_i', \ldots, P_k) = cost_i(P_1, \ldots, P_k) - cost_i(P_1, \ldots, P_i', \ldots, P_k),$$

for all strategy vectors $(P_1, \ldots, P_k)$ and for all player i and for all his or her strategies P_i', then f is a potential function of the game.

If a normal form game has a potential function, then we call it a potential game.

8 Existence of Pure Equilibria

In this section two points will be addressed. It will be demonstrated:

1. what benefit we gain from identifying a potential function in a game (Lemma 6), and
2. that all global-network-connection games have a potential function (Theorem 9) and therefore have a pure equilibrium (Corollary 10).

Addressing the first point, if we know that a game has a potential function, we can conclude that the game has a pure equilibrium. This conclusion can be drawn from the following Lemma:

Lemma 6. *Any local minimum of a potential function f is a pure equilibrium.*

Before we prove Lemma 6, we first define the notions of local and global minima as follows:

Definition 7. *We say* $(P_1^*, \ldots, P_k^*)$ *is a local minimum of* $f(P_1, \ldots, P_k)$ *if the replacement of one variable* P_i, $i \in [k]$, *cannot make* f *any smaller. In contrast,* $(P_1^*, \ldots, P_k^*)$ *is a global minimum of* f *if*

$$f(P_1^*, \ldots, P_k^*) \leq f(P_1, \ldots, P_k), \qquad \textit{for all } P_1, \ldots, P_k.$$

If we have a global minimum, it follows that it is also a local minimum.

Proof of Lemma 6. Consider a potential game with $f(P_1, \ldots, P_k)$ being its potential function. Also let $(P_1^*, \ldots, P_k^*)$ be a local minimum of f.

Now we show that $(P_1^*, \ldots, P_k^*)$ is a pure equilibrium of the game. Suppose player i selects a new strategy $P_i^{'}$, instead of P_i^*. By Definition 5, we have

$$f(P_1^*, \ldots, P_k^*) - f(P_1^*, \ldots, P_i^{'}, \ldots, P_k^*) = \mathrm{cost}_i(P_1^*, \ldots, P_k^*) - \mathrm{cost}_i(P_1^*, \ldots, P_i^{'}, \ldots, P_k^*).$$

However, since $(P_1^*, \ldots, P_k^*)$ is a local minimum of f, by Definition 7, we have

$$f(P_1^*, \ldots, P_k^*) \leq f(P_1^*, \ldots, P_i^{'}, \ldots, P_k^*),$$

and thus,

$$\mathrm{cost}_i(P_1^*, \ldots, P_k^*) \leq \mathrm{cost}_i(P_1^*, \ldots, P_i^{'}, \ldots, P_k^*).$$

This implies that $(P_1^*, \ldots, P_k^*)$ is a pure equilibrium of the game. □

Addressing the second point, we will show that all global-network-connection games have a potential function and therefore a pure equilibrium.

Definition 8. *For this proof, we define the function* f *of a global-network-connection game* G *as*

$$f(P_1, \ldots, P_k) = \sum_{e \in E} c_e \cdot \mathcal{H}_{k_e},$$

where c_e *is the cost of using edge* e*,* k_e *is the number of players using edge* e *in* $(P_1, \ldots, P_k)$*, and* $\mathcal{H}_{k_e}$ *is the* k_e*th harmonic number:*

$$\mathcal{H}_{k_e} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k_e}.$$

This leads us to our next Theorem.

Theorem 9. *Function* f *is a potential function.*

Corollary 10. *Showing that* f *is a potential function (Theorem 9) and that any local minimum of a potential function* f *is a pure equilibrium (Lemma 6) proves that all global-network-connection games have a pure equilibrium.*

Proof. In order to prove Theorem 9, we will check if our function f (Definition 8) is a potential function (Definition 5). Arranging our function f accordingly, we obtain:

$$f(P_1, \ldots, P_k) - f(P_1, \ldots, P_i', \ldots, P_k) = \sum_{e \in E} c_e \cdot \mathcal{H}_{k_e} - \sum_{e \in E} c_e \cdot \mathcal{H}_{k_e'} = \sum_{e \in E} c_e(\mathcal{H}_{k_e} - \mathcal{H}_{k_e'})$$

and

$$\text{cost}_i(P_1, \ldots, P_k) - \text{cost}_i(P_1, \ldots, P_i', \ldots, P_k) = \sum_{e \in P_i} \frac{c_e}{k_e} - \sum_{e \in P_i'} \frac{c_e}{k_e'},$$

where k_e is the number of players using e in $(P_1, \ldots, P_k)$ and k_e' is the number of players using e in $(P_1, \ldots, P_i', \ldots, P_k)$. We need to show that the two above are equal, which we will refer to as the left and right sides, respectively.

If we show for every edge $e \in E$ that its contribution to both sides are equal, then it follows that the two sides are equal and the requirement for potential functions (Definition 5) is satisfied.

For our case analysis, we let e be any edge in E. In total, we have four cases to consider:

1. If $e \in P_i$ and $e \in P_i'$,

   then trivially $k_e = k_e'$. Thus, the contribution to both sides is 0.

2. If $e \notin P_i$ and $e \notin P_i'$,

   then the argument is symmetric to case 1.

3. If $e \in P_i$ and $e \notin P_i'$,

   then $k_e = k_e' + 1$. Thus, the contribution to the left side is

$$\begin{aligned} & c_e(\mathcal{H}_{k_e} - \mathcal{H}_{k_e'}) \\ = \; & c_e((1 + \frac{1}{2} + \ldots + \frac{1}{k_e}) - (1 + \frac{1}{2} + \ldots + \frac{1}{k_e'})) \\ = \; & c_e((1 + \frac{1}{2} + \ldots + \frac{1}{k_e' + 1}) - (1 + \frac{1}{2} + \ldots + \frac{1}{k_e'}) \\ = \; & \frac{c_e}{k_e' + 1}, \end{aligned}$$

   which equals to its contribution to the right side.

4. If $e \notin P_i$ and $e \in P_i'$,

   then $k_e = k_e' - 1$ and the argument is similar to case 3.

Thus, we have demonstrated for every edge $e \in E$, its contribution to both sides are equal. Consequently, f is a potential function. $\square$

9 Proof of Upper Bound of Price of Stability

In this section we will prove an upper bound for the price of stability of global-network-connection games. Before we give the proof, we introduce definitions for the social optimum, price of stability, and price of anarchy.

Definition 11. *Given a global-network-connection game and any strategy vector* $(P_1, \dots, P_k)$*, we use*

$$\mathit{cost}(P_1, \dots, P_k) = \sum_{i \in [k]} \mathit{cost}_i(P_1, \dots, P_k) = \sum_{e \in P_1 \cup \dots \cup P_k} c_e$$

to denote its total cost. We say $(P'_1, \dots, P'_k)$ *is a social optimum (SO) if it minimizes the total cost:*

$$\mathit{cost}(P'_1, \dots, P'_k) \le \mathit{cost}(P_1, \dots, P_k), \qquad \text{for all } P_1, \dots, P_k.$$

Definition 12. *The price of stability (PoS) of a global-network-connection game is the ratio between the cost of a best (i.e., minimum) pure equilibrium (PE) and the cost of an SO:*

$$\frac{\min_{(P_1,\dots,P_k) \text{ is a PE}} \left\{\mathit{cost}(P_1, \dots, P_k)\right\}}{\mathit{cost}(P'_1, \dots, P'_k)},$$

where $(P'_1, \dots, P'_k)$ *is an SO of the game.*

The PoS is relevant for games in which there is some objective authority that can influence the players, and help them choose the best PE. Another important metric is the price of anarchy.

Definition 13. *The price of anarchy (PoA) of a game is the ratio between the cost of the worst (i.e., maximum) PE and the cost of an SO:*

$$\frac{\max_{(P_1,\dots,P_k) \text{ is a PE}} \left\{\mathit{cost}(P_1, \dots, P_k)\right\}}{\mathit{cost}(P'_1, \dots, P'_k)},$$

where $(P'_1, \dots, P'_k)$ *is an SO of the game.*

The PoA is used to measure how the efficiency of a system degrades due to selfish behavior of its players. From the above definitions it can be observed that the relationship between PoS, PoA, PE, and SO can be understood as follows:

$$\text{PoS} \quad \frac{\leftarrow \text{(best) PE (worst)} \rightarrow}{\text{SO}} \quad \text{PoA},$$

while we always have PoA $\ge$ PoS ≥ 1.

Theorem 14. *The PoS in the global connection game with* k *players is at most* $\mathcal{H}_k$*, the* k*th harmonic number (which is approximately equal to* $\log k$*).*

Proof. Let $(P_1, \dots, P_k)$ be any strategy vector. We show that

$$\text{cost}(P_1, \dots, P_k) \le f(P_1, \dots, P_k) \le \mathcal{H}_k \cdot \text{cost}(P_1, \dots, P_k).$$

Again we prove this by looking at each edge $e \in E$ and how much it contributes to $\text{cost}(P_1, \dots, P_k)$ and $f(P_1, \dots, P_k)$.

First, it can be seen that an edge e with k_e users contributes $c_e \cdot \mathcal{H}_{k_e}$ to the potential function f. Insofar, the potential function is an upper bound on the total cost $\text{cost}(P_1, \ldots, P_k)$ since $\mathcal{H}_{k_e} \geq 1$ whenever $k_e \geq 1$.

Second, we always have $\mathcal{H}_{k_e} \leq \mathcal{H}_k$ since we only have k players in total. This gives us the second part of the inequality above.

With this insight, we can now bound the PoS. Let $(P_1^{'}, \ldots, P_k^{'})$ be any SO of the game with cost $\text{cost}(P_1^{'}, \ldots, P_k^{'})$. We also let $(P_1^*, \ldots, P_k^*)$ denote a global minimum of the potential function f. The following inequalities hold:

$$\text{cost}(P_1^*, \ldots, P_k^*) \leq f(P_1^*, \ldots, P_k^*) \leq f(P_1^{'}, \ldots, P_k^{'}) \leq \mathcal{H}_k \cdot \text{cost}(P_1^{'}, \ldots, P_k^{'}).$$

Since $(P_1^*, \ldots, P_k^*)$ is a global minimum of f, it is also a local minimum, and thus a PE of the game. Because the best PE cannot be worse than the cost of the global minimum, this shows that $\mathcal{H}_k$ is an upper bound for the PoS of any global-network-connection game with k players. $\square$