COMS E6998: Algorithmic Game Theory and Economics — Feb 3, 2011

Lecture 3 – Network Congestion Game/Mixed (Nash) Equilibrium

Instructor: *Xi Chen* — Scribes: *Zheng Yifu* , *Liu Rui*

1 Network Congestion Game

Definition 1.1. *Review definition of Network Congestion Game:*

- *There're k players in the game*
- *$Player_i$ can pick any $path_i$ connecting s_i and t_i*
- $\forall e \in \mathbf{G}$: *cost function is $C_e(k_e)$*
  where, k_e is the number of players using the edge e in $(P_1, \ldots, P_k)$
- *The cost for $player_i$ is $C_i(P_1, \ldots, P_k) = \sum_{e \in P_i} C_e(k_e)$*
  where, k_e is the number of players using the edge e in $(P_1, \ldots, P_k)$
- *The total cost of the game is $C(P_1, \ldots, P_k) = \sum_{i=1}^{k} C_i(P_1, \ldots, P_k)$*

Theorem 1.1. *If all the $C_e(\cdot)$ are affine linear:*

$$C_e(k) = a_e * k + b_e \qquad (a_e, b_e \geq 0)$$

$$PoA \leq 2.618 = \frac{3+\sqrt{5}}{2}.$$

Proof. Let $(P_1^*, \ldots, P_k^*)$ be a PE and $(P_1', \ldots, P_k')$ be a social optimum:
The cost for $player_i$ in PE is:

$$\begin{aligned} C_i(P_1^*, \ldots, P_i^*, \ldots, P_k^*) &\leq C_i(P_1^*, \ldots, P_i', \ldots, P_k^*) \\ &= \sum_{e \in P_i'} C_e(\sharp \text{ players using } e \text{ in}(P_1^*, \ldots, P_i', \ldots, P_k^*)) \\ &\leq \sum_{e \in P_i'} C_e(k_e + 1) \\ &\leq \sum_{e \in P_i'} C_e(k_e + k_e') \end{aligned}$$

Then, we know the total cost of the game in PE is:

$$
\begin{aligned}
C(P_1^*,\dots,P_k^*) &= \sum_{i=1}^{k} C_i(P_1^*,\dots,P_k^*) \\
&\le \sum_{i=1}^{k} \sum_{e\in P_i'} C_e(k_e + k_e') \\
&= \sum_{e\in G} k_e' \cdot C_e(k_e + k_e') \\
&= \sum_{e\in G} k_e' \cdot (a_e \cdot (k_e + k_e') + b_e) \\
&= \sum_{e\in G} k_e' \cdot ((a_e \cdot k_e' + b_e) + a_e \cdot k_e) \\
&= \sum_{e\in G} k_e' \cdot (C_e(k_e') + a_e \cdot k_e) \\
&= \sum_{e\in G} k_e' \cdot C_e(k_e') + \sum_{e\in G} a_e \cdot k_e \cdot k_e' \\
&= C(P_1',\dots,P_k') + \sum_{e\in G} a_e \cdot k_e \cdot k_e'
\end{aligned}
$$

Next, let consider $\sum_{e\in G} a_e \cdot k_e \cdot k_e'$ (1):
By applying Cauchy Schwarz Inequality: for $x_1,\dots,x_n$ and $y_1,\dots,y_n$

$$(\sum_{i=1}^{n} x_i \cdot y_i)^2 \le (\sum_{i=1}^{n} x_i^2) \cdot (\sum_{i=1}^{n} y_i^2)$$

to (1), where, for $e \in G, x_e = \sqrt{a_e} \cdot k_e, y_e = \sqrt{a_e} \cdot k_e'$:

$$
\begin{aligned}
\sum_{e\in G} a_e \cdot k_e \cdot k_e' &= (\sum_{e\in G} a_e \cdot k_e \cdot k_e')^2 \\
&\le (\sum_{e\in G} a_e \cdot k_e^2)(\sum_{e\in G} a_e \cdot {k_e'}^2) \\
&\le C(P_1^*,\dots,P_k^*) \cdot C(P_1',\dots,P_k')
\end{aligned}
$$

The last step of derivation results from:

$$
\begin{aligned}
C(P_1',\dots,P_k') &= \sum_{e\in G} k_e' \cdot C_e(k_e') \\
&= \sum_{e\in G} k_e' \cdot (a_e \cdot k_e' + b_e) \\
&\ge \sum_{e\in G} a_e \cdot {k_e'}^2 \qquad \text{since, } a_e, b_e \ge 0
\end{aligned}
$$

Similarly, we can get $C(P_1^*, \ldots, P_k^*) \geq \sum_{e \in G} a_e \cdot k_e^2$ Thus, we have:

$$C(P_1^*, \ldots, P_k^*) \leq C(P_1', \ldots, P_k') + \sqrt{C(P_1^*, \ldots, P_k^*) \cdot C(P_1', \ldots, P_k')}$$

Divide the inequality above by $C(P_1', \ldots, P_k')$ on both sides:

$$\frac{C(P_1^*, \ldots, P_k^*)}{C(P_1', \ldots, P_k')} \leq 1 + \sqrt{\frac{C(P_1^*, \ldots, P_k^*)}{C(P_1', \ldots, P_k')}}$$

Denote $\alpha = \frac{C(P_1^*, \ldots, P_k^*)}{C(P_1', \ldots, P_k')}$:

$$\alpha \leq 1 + \sqrt{\alpha}$$

After deriving:

$$(\alpha - \frac{3}{2})^2 \leq \frac{5}{4}$$

Then, we get:

$$\alpha \leq \frac{3 + \sqrt{5}}{2}$$

Thus,

$$PoA = \frac{C(P_1^*, \ldots, P_k^*)}{C(P_1', \ldots, P_k')} \leq \frac{3 + \sqrt{5}}{2} = 2.618$$

The key point in the process of proving this theorem is to utilize the definition of PE to derive the relationship between $C(P_1^*, \ldots, P_k^*)$ and $C(P_1', \ldots, P_k')$. □

2 Mixed (Nash) Equilibrium

2.1 Introduction to Mixed Equilibrium

Consider a game $G = (\mathbf{A}, \mathbf{B}), where, \mathbf{A}, \mathbf{B}\ are\ both\ m \times n\ matrices:$

- There're two players in the game

- $player_1$ has m actions to choose from

- $player_2$ has n actions to choose from

- If $player_1$ picks $action_i$ and $player_2$ picks $action_j$

- $player_1$ receives $A_{i,j}$

- $player_2$ receives $B_{i,j}$

For example:

$$\mathbf{A} = \begin{array}{c} \\ \text{R} \\ \text{P} \\ \text{S} \end{array} \begin{array}{c} \begin{array}{ccc} R & P & S \end{array} \\ \begin{pmatrix} 0 & -1 & +1 \\ +1 & 0 & -1 \\ -1 & +1 & 0 \end{pmatrix} \end{array}$$

$$\mathbf{B} = \begin{array}{c} \\ \text{R} \\ \text{P} \\ \text{S} \end{array} \begin{array}{c} \begin{array}{ccc} R & P & S \end{array} \\ \begin{pmatrix} 0 & +1 & -1 \\ -1 & 0 & +1 \\ +1 & -1 & 0 \end{pmatrix} \end{array}$$

If $player_1$ picks $action_R$ and $player_2$ picks $action_S$, then, according to the matrices above, $player_1$ receives +1 and $player_2$ receives -1. At this time, $player_2$ must try to change to $action_P$, which is his or her best response to $player_1'$s $action_R$. Thus, we notice that, in this game, there's no PE if two players choose their own actions one by one.

Next, we modify the game a little bit like the following:

- $\triangle_\mathbf{m} = \{\mathbf{x} \in \mathrm{R}^\mathrm{m} \mid x_i \geq 0 \ \forall i \ and \ \sum_i x_i = 1\}$ is the probability distribution set of $player_1'$s actions
- $\triangle_\mathbf{n} = \{\mathbf{x} \in \mathrm{R}^\mathrm{n} \mid y_i \geq 0 \ \forall i \ and \ \sum_i y_i = 1\}$ is the probability distribution set of $player_2'$s actions
- $player_1$ has m actions to choose from in any action's probability distribution $\mathbf{x} \in \triangle_\mathbf{m}$
- $player_2$ has n actions to choose from in any action's probability distribution $\mathbf{y} \in \triangle_\mathbf{n}$
- $player_1$ picks $action_i$ with probability x_i
- $player_2$ picks $action_j$ with probability y_j

Definition 2.1.1. (i, j) *is a pure equilibrium if:*

1. *$action_i$ is a best response w.r.t. $action_j$, if $A_{i,j} \geq A_{i',j}, \forall i' \in [m]$*
2. *$action_j$ is a best response w.r.t. $action_i$, if $B_{i,j} \geq B_{i,j'}, \forall j' \in [n]$*

Definition 2.1.2. $(\mathbf{x}, \mathbf{y})$ $(\mathbf{x} \in \triangle_\mathbf{m}, \mathbf{y} \in \triangle_\mathbf{n})$ *is a mixed equilibrium if:*

1. *$\mathbf{x}$ is a best response w.r.t. $\mathbf{y}$*
   $\mathbf{x^T A y} \geq (\mathbf{x'})^\mathbf{T} \mathbf{A y}, \forall \mathbf{x'} \in \triangle_\mathbf{m}$
   where, $\mathbf{x}^T \mathbf{A y} = \sum_{ij} x_i A_{i,j} y_j$, which is the expected payoff
2. *$action_y$ is a best response w.r.t. $action_x$*
   $\mathbf{x^T B y} \geq \mathbf{x^T B}(\mathbf{y'}), \forall \mathbf{y'} \in \triangle_\mathbf{n}$

2.2 Advantages and Disadvantages of Mixed Equilibrium

stable	there may be multiple/infinite many mixed equilibria
always exit (Nash 1950)	do not consider risk
	inefficient
	different from experiments

3 Two-play zero sum games

3.1 Definition of the game

In a two-player zero sum game, we are given a $m \times n$ matrix $\mathbf{M}$ and two players. G=($\mathbf{M}, -\mathbf{M}$). Therefore, there are m strategies for player 1 and n strategies for player 2. In this game, Player 1 picks a m-dimensional distribution $\mathbf{x} \in \mathbf{\Delta_m}$, while player 2 picks a n-dimensional distribution $\mathbf{y} \in \mathbf{\Delta_n}$. According to the distribution $\mathbf{x}$ *and* $\mathbf{y}$, player 1 picks the strategy i with the probability x_i and player 2 picks the strategy j with the probability y_j. And if they do so, player 1 will get the utility of $M_{i,j}$, while player 2 will pay cost of $-M_{i,j}$. The sum of the payoffs of the two players is zero as the name of the game indicates.

Definition 3.1.1. *($\mathbf{x}, \mathbf{y}$) is a mixed equilibrium in a two-player zero sum game if it satisfies the following inequalities,*

$$\forall \mathbf{x}' \in \mathbf{\Delta_m},\ \mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{y} \geq (\mathbf{x}')^{\mathrm{T}}\mathbf{M}\mathbf{y}$$

$$\forall \mathbf{y}' \in \mathbf{\Delta_n},\ \mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{y} \leq \mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{y}'$$

von Neumann's proof later implies a polynomial time algorithm for finding a mixed equilibrium.

3.2 Existence of mixed equilibrium

In order to prove the existence of mixed equilibrium in the two-player zero sum games, first of all we introduce the Minimax Theorem.

Theorem 3.2.1. $max_{\mathbf{x}\in\mathbf{\Delta_m}}(min_{\mathbf{y}\in\mathbf{\Delta_n}}\mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{y}) = min_{\mathbf{y}\in\mathbf{\Delta_n}}(max_{\mathbf{x}\in\mathbf{\Delta_m}}\mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{y})$*, where x is a m-dimensional distribution, y is a n-dimensional distribution and M is a $m \times n$ matrix.*

To see why this theorem will be helpful in proving the existence of mixed equilibrium in the two-player zero sum games, let's consider the following two cases.
Case 1: Player 1 picks $\mathbf{x} \in \mathbf{\Delta_m}$ first.

In this case, player 2 will pick $\mathbf{y} \in \mathbf{\Delta_n}$ that makes $\mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{y} = min_{\mathbf{y}'\in\mathbf{\Delta_n}}\mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{y}'$. To make the utility maximum, player 1 of course picked $\mathbf{x} \in \mathbf{\Delta_m}$ in the way that,

$$\mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{y} = max_{\mathbf{x}'\in\mathbf{\Delta_m}}(min_{\mathbf{y}'\in\mathbf{\Delta_n}}(\mathbf{x}')^{\mathrm{T}}\mathbf{M}\mathbf{y}') \tag{1}$$

Case 2: Player 2 picks $\mathbf{y} \in \mathbf{\Delta n}$ first.

In this case, player 1 will pick $\mathbf{x} \in \mathbf{\Delta m}$ that makes $\mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{y} = max_{\mathbf{x}'\in\mathbf{\Delta_m}}(\mathbf{x}')^{\mathrm{T}}\mathbf{M}\mathbf{y}$. To make the cost miminum, player 2 picked $\mathbf{y} \in \mathbf{\Delta_n}$ in the way that,

$$\mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{y} = min_{\mathbf{y}'\in\mathbf{\Delta n}}(max_{\mathbf{x}'\in\mathbf{\Delta_m}}(\mathbf{x}')^{\mathrm{T}}\mathbf{M}\mathbf{y}') \tag{2}$$

In result, $(\mathbf{x}, \mathbf{y})$ is a mixed equilibrium is equivalent to that the right side of equation (1) and the right side of equation (2) are equal to each other.

Now we make use of Minimax Theorem to prove the existence of mixed equilibrium in the two-player zero sum games.

Proof. Denote $\mathbf{x}^* = argmax_{\mathbf{x}\in\mathbf{\Delta m}}(min_{\mathbf{y}\in\mathbf{\Delta n}}\mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{y})$, $\mathbf{y}^* = argmin_{\mathbf{y}\in\mathbf{\Delta_n}}(max_{\mathbf{x}\in\mathbf{\Delta m}}\mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{y})$. Now we prove that $(\mathbf{x}^*, \mathbf{y}^*)$ is a mixed equilibrium.

As the definitions of $\mathbf{x}^*$ and $\mathbf{y}^*$ indicates,

$$(\mathbf{x}^*)^{\mathrm{T}}\mathbf{M}\mathbf{y}^* \leq max_{\mathbf{x}\in\mathbf{\Delta m}}\mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{y}^* \tag{3}$$

$$(\mathbf{x}^*)^{\mathrm{T}}\mathbf{M}\mathbf{y}^* \geq min_{\mathbf{y}\in\mathbf{\Delta n}}(\mathbf{x}^*)^{\mathrm{T}}\mathbf{M}\mathbf{y} \tag{4}$$

And according to Minimax Theorem, we have that,

$$max_{\mathbf{x}\in\mathbf{\Delta m}}\mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{y}^* = min_{\mathbf{y}\in\mathbf{\Delta n}}(\mathbf{x}^*)^{\mathrm{T}}\mathbf{M}\mathbf{y} \tag{5}$$

Define $V = max_{\mathbf{x}\in\mathbf{\Delta m}}\mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{y}^* = min_{\mathbf{y}\in\mathbf{\Delta n}}(\mathbf{x}^*)^{\mathrm{T}}\mathbf{M}\mathbf{y}$.

So, with respect to inequality (3), (4) and the equation (5), we get that,

$$(\mathbf{x}^*)^{\mathrm{T}}\mathbf{M}\mathbf{y}^* = V = max_{\mathbf{x}\in\mathbf{\Delta_m}}(min_{\mathbf{y}\in\mathbf{\Delta_n}}\mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{y}) = min_{\mathbf{y}\in\mathbf{\Delta_n}}(max_{\mathbf{x}\in\mathbf{\Delta_m}}\mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{y}) \tag{6}$$

Hence, we prove the existence of mixed equilibrium in the two-player zero sum games. □

3.3 Proof of Minimax Theorem

In section 3.2, we made use of Minimax Therom without proving. Now we will prove this theorem. Before that, we will first introduce another theorem, named as Duality Theorem.

Theorem 3.3.1. *LP1 and LP2 are two linear programs which are dual to each other. The two programs are defined as follows.*
LP1:

$$maximize\ \mathbf{c}^{\mathrm{T}}\mathbf{x} = (c_1\ c_2\ \dots\ c_k)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} = opt1 \tag{7}$$

subject to:

$$\mathbf{A}\mathbf{x} \leq \mathbf{b} \tag{8}$$

$$\mathbf{x} \geq \mathbf{0} \tag{9}$$

where $\mathbf{A}$ *is a* $l \times k$ *matrix and* $\mathbf{b}$ *is a* $l \times 1$ *vector.*
LP2:

$$minimize\ \mathbf{b}^{\mathrm{T}}\mathbf{y} = (b_1\ b_2\ \dots\ b_l)\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_l \end{pmatrix} = opt2 \tag{10}$$

subject to:

$$\mathbf{A}^{\mathrm{T}}\mathbf{y} \geq \mathbf{c} \tag{11}$$

$$\mathbf{y} \geq \mathbf{0} \tag{12}$$

where $\mathbf{A}, \mathbf{b}, \mathbf{c}$ *are all the same as those in LP1.*

Considering LP1 and LP2 defined above, we have,

$$opt1 = opt2 \tag{13}$$

Next, we denote $\mathbf{M_{*,j}}$ as the j^{th} column of matrix $\mathbf{M}$ and $\mathbf{M_{i,*}}$ as the i^{th} row of matrix $\mathbf{M}$. In this way, we have,

$$min_{\mathbf{y}}\mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{y} = min_{j\in[n]}\mathbf{x}^{\mathrm{T}}\mathbf{M_{*,j}} \tag{14}$$

$$max_{\mathbf{x}}\mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{y} = max_{i\in[m]}\mathbf{M_{i,*}}\mathbf{y} \tag{15}$$

In this way, we could rewrite the Minimax Theorem as follows,

$$max_{\mathbf{x}\in\mathbf{\Delta_m}}(min_{j\in[n]}\mathbf{x}^{\mathrm{T}}\mathbf{M_{*,j}}) = min_{\mathbf{y}\in\mathbf{\Delta_n}}(max_{i\in[m]}\mathbf{M_{i,*}}\mathbf{y}) = V \tag{16}$$

Now the proof of the Minimax Theorem is as follows,

Proof. Define LP1 as:

$$maximize\ v \tag{17}$$

subject to:

$$\mathbf{x} \geq \mathbf{0} \tag{18}$$

$$\Sigma x_i = 1 \tag{19}$$

$$\forall j \in [n], \mathbf{x}^{\mathrm{T}}\mathbf{M_{*,j}} \geq v \tag{20}$$

Define LP2 as:

$$minimize\ w \tag{21}$$

subject to:

$$\mathbf{y} \geq \mathbf{0} \tag{22}$$

$$\Sigma y_i = 1 \tag{23}$$

$$\forall i \in [m], \mathbf{M_{i,*}}\mathbf{y} \leq w \tag{24}$$

Then according to Duality Theorem, We have

$$max_{\mathbf{x}\in\mathbf{\Delta_m}}(min_{j\in[n]}\mathbf{x}^{\mathrm{T}}\mathbf{M_{*,j}}) = min_{\mathbf{y}\in\mathbf{\Delta_n}}(max_{i\in[m]}\mathbf{M_{i,*}}\mathbf{y}) \tag{25}$$

which is equivalent to Minimax Theorem. □