COMS E6998: Algorithmic Game Theory and Economics Mar 10, 2011

Lecture 8 – Arrow-Debreu Exchange Market and Fisher's Model

Instructor: *Xi Chen* Scribes: *Yi Zhang, Minglei Wang*

1 Arrow-Debreu Exchange Market

Last week, we talked about the definition about Arrow-Debreu Exchange Market:

Definition 1 (Arrow-Debreu Exchange Market). *An Arrow-Debreu Exchange Market is characterized by the following components:*

1. *Traders: 1, 2, ..., k;*
2. *Goods: 1, 2, ..., n;*
3. *Each trader has an initial endowment $\mathbf{e}_i \in \mathbb{R}^n_+$, e_{ij} unit of Good j. We can assume without loss of generality that for every good $j \in [n]$,*
$$\sum_{i=1}^{k} e_{ij} = 1;$$
4. *Each trader also has a utility function $u_i : \mathbb{R}^n_+ \to \mathbb{R}_+$ and wants to maximize his own utility;*

Suppose each good j has a price $p_j \geq 0$. Without loss of generality, we assume the price vector $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ satisfies $\sum_{j=1}^{n} p_j = 1$. We give the definition of optimal bundles:

Definition 2 (Optimal Bundle). *Given $\mathbf{p} \in \Delta_n$ and $b_i = \mathbf{e}_i \cdot \mathbf{p} = \sum_{j \in [n]} e_{ij} \cdot p_j$, then $\mathbf{x}_i \in \mathbb{R}^n_+$ is an optimal bundle for trader i if the following two conditions hold:*

1. **Budget Constraint** $\mathbf{x}_i \cdot \mathbf{p} \leq b_i$,
2. **Optimality** $u_i(\mathbf{x}_i) \geq u_i(\mathbf{x})$, $\forall \mathbf{x} \in \mathbb{R}^n_+$ *such that* $\mathbf{x} \cdot \mathbf{p} \leq b_i$.

We can show that if u_i is strictly concave and has an optimal bundle $\mathbf{x}_i$, then $\mathbf{x}_i$ is the only optimal bundle.

Lemma 3 (Uniqueness). *Let $\mathbf{p} \in \Delta_n$ be a price vector. If u_i is strictly concave and $\mathbf{x}_i$ is an optimal bundle for trader i, then it is the only optimal bundle for trader i.*

However, in general it is not true that an optimal bundle always exists. For example, if the price of good 1 is zero and the utility function for trader 1 is $u_1(x_1, x_2, \ldots, x_n) = x_1$, there is no optimal bundle for this trader. Because if trader 1 want to maximize his utility, he would buy good 1 as much as he can. However, good 1 is free in this case, the demand of trader 1 can never be satisfied. Thus the trader has no optimal bundle.

We now introduce a notion, restrictively optimal bundle, which is slightly different from optimal bundles but always exists:

Definition 4 (Restrictively Optimal Bundle)**.** *Given* $\mathbf{p} \in \Delta_n$*,* $\mathbf{x}_i$ *is an restrictively optimal bundle for trader* i *if* $\mathbf{x}_i$ *is the best in* $[0, 1.1]^n$*, that is:*

1. **Budget Constraint** $\mathbf{x}_i \cdot \mathbf{p} \le b_i = \mathbf{e}_i \cdot \mathbf{p}$*,*
2. **Optimality** $u_i(\mathbf{x}_i) \ge u_i(\mathbf{x})$*,* $\forall \mathbf{x} \in [0, 1.1]^n$ *such that* $\mathbf{x} \cdot \mathbf{p} \le b_i$*.*

For this new definition, there are two nice properties:

- It always exists and is unique;
- Even though it is restricted, i.e., optimal only within the bounded hypercube $[0, 1.1]^n$, there is a strong connection with the (global) optimal bundle — if $\mathbf{x}_i$ is a restrictively optimal bundle for trader i and $x_{ij} \le 1$ for all $j \in [n]$, it is an optimal bundle.

We have a corollary from the second property:

Corollary 5. *If we can find* $\mathbf{p} \in \Delta_n$ *and* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ *such that* $\mathbf{x}_i$ *is restrictively optimal and satisfies the market clearance condition* $\sum_{i=1}^{k} \mathbf{x}_i = \mathbf{1}_n$*, then it is a market equilibrium.*

This is very useful because to prove the existence of a market equilibrium, we now only need to find a price vector $\mathbf{p}$ together with restrictively optimal bundles $\mathbf{x}_1, \ldots, \mathbf{x}_k$ which clear the market.

Let's define a map: $\mathbf{x}_i(\mathbf{p})$ is the restrictive optimal bundle for trader i with respect to price vector $\mathbf{p}$. It is guaranteed to exist and is unique by the first property.

It is very important to have $\mathbf{x}$ in a bounded space in the definition of restrictively optimal bundles, which enables the application of Maximum Theorem. It follows that $\mathbf{x}_i(\mathbf{p})$, as a map over Δ_n, is continuous. The continuity of $\mathbf{x}_i(\mathbf{p})$ is important because we will use these maps to construct a continuous map f to apply Brouwer's Fixed Point Theorem. We will prove the existence of a market equilibrium by showing that any fixed point of f is a market equilibrium.

Proof: The existence of a market equilibrium. To this end, we need to define a few concepts.

Recall that given any $\mathbf{p} \in \Delta_n$, $\mathbf{x}_i(\mathbf{p})$ is *the* restrictively optimal bundle of trader i. We define the *aggregate excess demand* to be

$$\mathbf{z}(\mathbf{p}) = \sum_{i=1}^{k} \mathbf{x}_i(\mathbf{p}) - \mathbf{1}_n.$$

Remember the initial endowments satisfy for every good $j \in [n]$,

$$\sum_{i=1}^{k} e_{ij} = 1.$$

thus the total supply for good j is 1. And $\sum_{i=1}^{k} \mathbf{x}_i(\mathbf{p})$ is the total demand, the difference between total demand and supply is the aggregate excess demand.

We now define a function $\mathbf{f} : \Delta_n \to \Delta_n$. The intuition behind the definition is the following: We set $f(\mathbf{p}) = \mathbf{p}'$ in a way such that for each good $j \in [n]$,

- If $z_j(\mathbf{p}) > 0$, it indicates demand is strictly higher than supply, so we should raise the price;
- If $z_j(\mathbf{p}) < 0$, it indicates demand is strictly lower than supply, so we should lower the price; and
- If $z_j(\mathbf{p}) = 0$ all equal to 0, then this is a market equilibrium.

Then, we can define f as follows: Given any $\mathbf{p} \in \Delta_n$, we set $\mathbf{p'} = f(\mathbf{p})$ where

$$p'_j = \frac{p_j + \max(0, z_j(\mathbf{p}))}{1 + \sum_{\ell=1}^{n} \max(0, z_\ell(\mathbf{p}))}, \qquad \text{for all } j \in [n],$$

It basically works as follows: if $z_j(\mathbf{p})$ is positive, then f will raise the price of good j; if $z_j(\mathbf{p})$ is negative, f will lower the price of good j.

We can learn from this mapping that

1. The prices sum to one, $\sum_{j=1}^{n} p'_j = 1$, so f is indeed a map from Δ_n to itself;
2. The map f is continuous because all the $\mathbf{x}_i(\cdot)$'s are continuous;
3. We show below that if there exists a $\mathbf{p}^*$ that $f(\mathbf{p}^*) = \mathbf{p}^*$, then this is a market equilibrium.

Look at the third one, that's because: If $f(\mathbf{p}^*) = \mathbf{p}^*$, we will have

$$p^*_j = \frac{p^*_j + \max(0, z_j(\mathbf{p}^*))}{1 + \sum_{\ell=1}^{n} \max(0, z_\ell(\mathbf{p}^*))}.$$

And thus we have $\forall j \in [n]$, $z_j(\mathbf{p}^*) \leq 0 \Rightarrow \sum_{i=1}^{k} x_{ij}(\mathbf{p}^*) \leq 1$, which reads: the total demand of each good is at most one. We leave it as an exercise to show that this actually implies that the demand is equal to the supply:

$$\sum_{i=1}^{k} x_{ij}(\mathbf{p}^*) = 1, \qquad \text{for all } j \in [n].$$

Now we finish the proof. Because the map f is continuous and is from Δ_n to itself, it has a fixed point that $f(\mathbf{p}^*) = \mathbf{p}^*$. By the discussion above, it indicates that $\mathbf{x}_1(\mathbf{p}^*), \ldots, \mathbf{x}_k(\mathbf{p}^*)$ are restrictively optimal bundles of the traders that clear the market. By Corollary 5, we know $\mathbf{p}^*$ is a market equilibrium. □

2 Fisher's Model with Linear Utilities

Here, we introduce Fisher's model with linear utilities which are not strictly concave.

In this simpler model, goods do not come from traders. At the beginning the market has 1 unit of each good. We also don't have traders but just buyers in this market, i.e. there is no selling going on. Each buyer has a budget and a utility function which is linear.

Definition 6 (Fisher's Model)**.** *A market in Fisher's Model can be characterized by the following components:*

1. *Buyers:* $1, 2, \ldots, k$*;*
2. *Goods:* $1, 2, \ldots, n$*;*
3. *Each buyer has a budget:* $b_i > 0$*,* $i \in [k]$*;*
4. *Each buyer has a linear utility function:* $u_i(\mathbf{x}_i) = u_i(x_{i1}, x_{i2}, \ldots, x_{in}) = \sum_{j=1}^{n} u_{ij} \cdot x_{ij}$*,* $u_{ij} \geq 0$*.*

Again we define optimal bundles.

Definition 7 (Optimal Bundle)**.** *Given prices for the goods* $\mathbf{p} = (p_1, p_2, \ldots, p_n)$*, where* $p_j \geq 0$*,* $\mathbf{x}_i$ *is an optimal bundle for buyer* i *if*

1. **Budget Constraint** $\mathbf{x}_i \cdot \mathbf{p} \leq b_i$*; and*

2. **Optimality** $u_i(\mathbf{x}_i) \geq u_i(\mathbf{x})$*, for all* $\mathbf{x}$ *such that* $\mathbf{x} \cdot \mathbf{p} \leq b_i$*.*

Definition 8 (Market Equilibrium)**.** *A price vector* $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ *is a market equilibrium if there exist* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in \mathbb{R}^n_+$ *such that for all* $i \in [k]$*,* $\mathbf{x}_i$ *is optimal for buyer* i *with respect to* $\mathbf{p}$*, and the demand equals the supply*

$$\sum_{i=1}^{k} \mathbf{x}_i = \mathbf{1}_n.$$

For this model, we can have following theorem

Theorem 9 (Existence of Equilibrium)**.** *If every good has a potential buyer, every market in Fisher's model has an equilibrium.*

Buyer i is a potential buyer of good j if $u_{ij} > 0$. If $u_{ij} = 0$ buyer i has no interest in buying good j. We will use Eisenberg-Gale convex program [1959] to prove this theorem. We define the program as:

- Variables: x_{ij}, where $i \in [k]$ and $j \in [n]$;
- Maximizes the following function
$$\sum_{i=1}^{k} b_i \cdot \log\left(\sum_{j=1}^{n} u_{ij} x_{ij}\right);$$
- Subject to
$$\sum_{i=1}^{k} x_{ij} \leq 1, \qquad \text{for all } j \in [n]$$
(the demand never exceeds 1) and
$$x_{ij} \geq 0, \qquad \text{for all } i \in [k] \text{ and } j \in [n].$$

This is a convex program since the function we try to maximize is concave. It has two implications:

1. The set of optimal solutions is convex; and
2. We can solve it very efficiently. If we have the connection of optimum solution here, to the market equilibrium, we get a very efficient algorithm.

To prove the theorem, we use the KKT condition to show that every optimal solution of the Eisenberg-Gale convex program corresponds to a market equilibrium. Once we plug in the KKT condition to the convex program, we would get a lot of inequalities which guarantee that the x_{ij}'s would give us a market equilibrium of this Fisher's model market.

First we look at a simple problem to get some idea of the KKT condition. If we have two variables x_1 and x_2 and we want to maximize $f(x_1, x_2) = x_1 + x_2$ which is subject to $g(x_1, x_2) = x_1^2 + x_2^2 \leq 2$.

From Figure 1, we know that all the possible solutions should lie in the disk, and the intersection of line $y = x$ and the circle at $(1, 1)$ should be the optimal solution.

However, a formal proof of this optimal solution is much more complicated as followed.

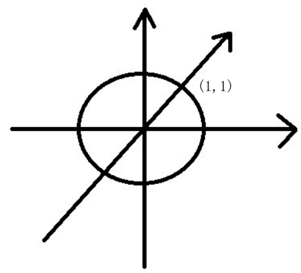

Figure 1: Optimal Solution

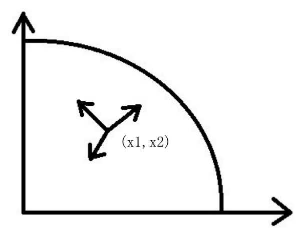

Figure 2: Case 1

Proof. Suppose $(1, 1)$ is not the optimal solution. So, let's look at some points different.

Case 1: (x_1, x_2) is optimum and lies inside the disk.

If (x_1, x_2) is really optimum, then moving the point along any direction cannot increase the objective function. Now we have such fact that for

$$\nabla f(x_1, x_2) := \left(\frac{\partial f(x_1, x_2)}{\partial x_1}, \frac{\partial f(x_1, x_2)}{\partial x_2} \right)$$

we have

$$\text{If } \mathbf{v} \neq \mathbf{0} \text{ satisfies } \mathbf{v} \cdot \nabla f(x_1, x_2) > 0, \text{ then } f(\mathbf{x} + \epsilon \mathbf{v}) > f(\mathbf{x}) \text{ for small enough } \epsilon > 0,$$

where we use $\mathbf{x}$ to denote (x_1, x_2). This implies that $\nabla f(x_1, x_2) = \mathbf{0}$.

Case 2: (x_1, x_2) lies on the boundary: $g(x_1, x_2) = 2$. We make the following claim.

Claim 10. *If we can find a nonzero vector* $\mathbf{v}$ *such that*

$$\mathbf{v} \cdot \nabla f(x_1, x_2) > 0 \quad \textit{and} \quad \mathbf{v} \cdot \nabla g(x_1, x_2) < 0,$$

it also proves that (x_1, x_2) *is not optimal.*

Proof. If we move the point (x_1, x_2) along the direction $\mathbf{v}$ by a little bit, the $g(\cdot)$ function would decrease and it can only go inside the circle, so $\mathbf{x} + \epsilon \mathbf{v}$ is feasible for small enough $\epsilon > 0$. On the other hand, we also know that moving (x_1, x_2) along $\mathbf{v}$ by a little bit, we can increase the objective function f. Hence, it implies that (x_1, x_2) can not be optimal. □

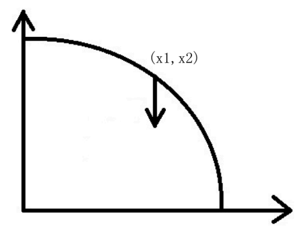

Figure 3: Case 2

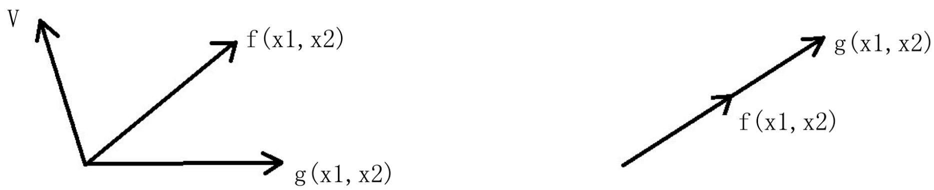

Figure 4: Vector Direction

Assume we have two nonzero vectors $\nabla f(x_1, x_2)$ and $\nabla g(x_1, x_2)$, to avoid the situation above, these two vectors have to be in the same direction. Otherwise the claim above would imply that (x_1, x_2) is not optimal.

The discussion above provides us some ideas behind the KKT conditions, a set necessary conditions that an optimal solution must satisfy. For the simple example above, by applying the KKT condition we get the following necessary conditions for (x_1, x_2) being optimal:

1. There exists a $\mu \geq 0$ such that $\nabla f(x_1, x_2) = \mu \cdot \nabla g(x_1, x_2)$; and
2. If $g(x_1, x_2) < 2$ then $\mu = 0$.

This is essentially a more concise way to present our discussion of the two cases above.

For example, we can use the KKT conditions to formally prove that $(1, 1)$ is the only optimum. Suppose (x_1, x_2) is optimal, then we plug in these two conditions. First of all, there exists a $\mu \geq 0$ that

$$(1, 1) = \mu \cdot (2x_1, 2x_2)$$

so $x_1 = x_2$. Also, if $x_1^2 + x_2^2 < 2$, then $\mu = 0$. But by the equation above μ cannot be 0. So the only point (x_1, x_2) that satisfies both conditions is $(1, 1)$. $\square$

And then let's go down to the general case. We define the program as

- Variables: $x_1, \ldots, x_n$;
- Maximizes $f(x_1, \ldots, x_n)$;

- Subject to

$$
\begin{aligned}
g_1(x_1, \ldots, x_n) &\le 0 \\
g_2(x_1, \ldots, x_n) &\le 0 \\
&\vdots \\
g_s(x_1, \ldots, x_n) &\le 0
\end{aligned}
$$

The KKT conditions (for $\mathbf{x} = (x_1, \ldots, x_n)$ being an optimal solution) are:

There exist $\mu_1, \mu_2, \ldots, \mu_s \ge 0$ such that

- $\triangledown f(\mathbf{x}) = \mu_1 \cdot \triangledown g_1(\mathbf{x}) + \cdots + \mu_s \cdot \triangledown g_s(\mathbf{x})$; and
- For every $i \in [s]$, if $g_i(\mathbf{x}) < 0$ then $\mu_i = 0$.

Finally we apply these conditions to the Eisenberg-Gale convex program. Assume x_{ij} is an optimal solution, then it should satisfy the following conditions: (Here we use $p_1, \ldots, p_n$ to denote the multipliers for inequalities

$$\sum_{i=1}^{k} x_{i1} \le 1, \ldots, \sum_{i=1}^{k} x_{in} \le 1,$$

respectively. These multipliers in KKT turn out to be the prices in the market equilibrium.)

1. $\sum_{i=1}^{k} x_{ij} \le 1$ for all $j \in [n]$ and $x_{ij} \ge 0$ for all $i \in [k]$ and $j \in [n]$;
2. $p_j \ge 0$ for all $j \in [n]$;
3. For every $i \in [k]$ and $j \in [n]$,
$$p_j \ge \frac{b_i \cdot \mu_{ij}}{\sum_{k=1}^{n} u_{ik} \cdot x_{ik}}$$
4. For every $j \in [n]$, $p_j > 0 \Rightarrow \sum_{i=1}^{k} x_{ij} = 1$; and
5. For every $i \in [k]$ and $j \in [n]$, $x_{ij} > 0$ implies
$$p_j = \frac{b_i \cdot u_{ij}}{\sum_{k=1}^{n} u_{ik} \cdot x_{ik}}.$$

Proof. We use the KKT conditions to show that $(p_1, \ldots, p_n)$ together with $\mathbf{x}_i = (x_{i1}, \ldots, x_{in})$ give us a market equilibrium.

Step 1: First, we show that $p_j > 0$ for all $j \in [n]$.

This is because every good has at least one potential buyer, i.e. for all j, we can find a buyer i that $u_{ij} > 0$. Then, from condition 3, we can find out that $p_j > 0$.

Step 2: Since $p_j > 0$, by condition 4, we know that $\sum_{i=1}^{k} x_{ij} = 1$.

The only thing left is to show that $\mathbf{x}_i = (x_{i1}, \ldots, x_{in})$ is optimal for buyer i.

Because the utility function is linear,

$$u_i(\mathbf{x}) = u_{i1} \cdot x_1 + u_{i2} \cdot x_2 + \ldots + u_{in} \cdot x_n.$$

For buyer i, the budget is $b_i > 0$. And given the price of goods $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, in which $p_j > 0$ for all j, the buyer tries to maximize the utility. So, the buyer would buy the good which makes the most contribution to the utility function per unit of price. We define bang-per-buck of good j to be:

$$\frac{u_{ij}}{p_j}$$

and we let

$$\mathrm{bpb}_i = \max_{j \in [n]} \frac{u_{ij}}{p_j}.$$

Consider condition 3, for every i and j,

$$\sum_{k=1}^{n} u_{ik} \cdot x_{ik} \geq b_i \cdot \frac{u_{ij}}{p_j}.$$

So, for all j, the total utility is $\geq b_i \cdot \frac{u_{ij}}{p_j}$. It means that the total utility is $\geq b_i \cdot \mathrm{bpb}_i$. The right-hand side implies that bundle $\mathbf{x}_i$ is an optimal bundle for buyer i, because the buyer spends all money on goods that give highest bang-per-buck.

In summary, we have shown that if $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ is an optimal solution to the Eisenberg-Gale convex program and $(p_1, \ldots, p_n)$ are multipliers from the KKT conditions, then $(p_1, \ldots, p_n)$ is a market equilibrium where $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are optimal bundles that clear the market. As a result, we can solve this convex program efficiently to find a market equilibrium. $\square$