COMS E6998: Algorithmic Game Theory and Economics Feb 10, 2011

Lecture 4 – Nash's Mixed Equilibrium and Brouwer's Fixed Point

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1 Nash Equilibrium

Last week, we showed that every zero-sum, two-player game has a mixed equilibrium [Von Neumann, 1929]. This week, we will prove that every game has a mixed equilibrium [Nash, 1950].

To succinctly prove the existence of this mixed equilibrium, we will make use of Brouwer's fixed point theorem:

Theorem 1 (Brouwer's Fixed Point Theorem)**.** *Given $S \subset \mathbb{R}^n$, where S is convex, bounded, and closed, every continuous map $f : S \to S$ has a fixed point $x^* : f(x^*) = x^*$.*

1.1 Real-world Experiment: A Map

In order to better grasp the concept of Brouwer's Theorem we will start by a real-world experiment. We start by assuming that we have two copies of the same map, each with different scales. Every point on the map corresponds to a geographic place. We now place the smaller map anywhere (and at any angle) on the larger map. We say x is a magic point if the geographic place it corresponds on the smaller map is exactly the same as the place it corresponds on the larger one.

Brouwer's fixed point theorem then claims that no matter where we place the smaller map, as long as it is completely inside the bigger map, we will always be able to find a magic point.

1.2 Existence of Nash Equilibria

Now we will use the Fixed Point Theorem to prove the existence of Nash equilibria. We begin with the following definitions: We let $G = (A, B)$ denote a two-player game, and let $\mathbf{x} \in \Delta_m$ denote a mixed strategy of player 1 and $\mathbf{y} \in \Delta_n$ denote a mixed strategy of player 2.

Definition 2. *We define $(\mathbf{x}, \mathbf{y})$ to be a mixed equilibrium if $\mathbf{x}$ is a best-response with respect to $\mathbf{y}$, and $\mathbf{y}$ is a best-response with respect to $\mathbf{x}$.*

Definition 3. *There are three equivalent definitions of best-response:*

1. $\mathbf{x}^T\mathbf{A}\mathbf{y} \geq (\mathbf{x}')^T\mathbf{A}\mathbf{y}$, *for all* $\mathbf{x}' \in \Delta_m$;
2. $\mathbf{x}^T\mathbf{A}\mathbf{y} = \max_{i\in[m]} \mathbf{A}_{i,*}\mathbf{y}$;
3. $x_k > 0 \;\Rightarrow\; \mathbf{A}_{k,*} = \max_{i\in[m]} \mathbf{A}_{i,*}\mathbf{y}$.

We want to be able to define a continuous map f from G, which maps any pair $(\mathbf{x}, \mathbf{y})$ of probability distributions to a new pair $(\mathbf{x}', \mathbf{y}')$ of probability distributions: $f : \Delta_m \times \Delta_n \to \Delta_m \times \Delta_n$, and use it together with Brouwer's fixed point theorem to prove the existence of Nash equilibria in G. We have proven Nash's Theorem if we can show that the map has the following properties:

1. It must be continuous.

2. If $(\mathbf{x}, \mathbf{y})$ is a fixed point of f, then it is a mixed equilibrium of G.

We begin with the definition of f and proof of the first property.

Proof. If $(\mathbf{x}, \mathbf{y})$ is already a Nash Equilibrium, we will map it to itself. If $\mathbf{x}$ is not a best response, then we replace it with a new $\mathbf{x}'$. Similarly, if $\mathbf{y}$ is not a best response, we replace it with $\mathbf{y}'$.

Our strategy will be to decrease the probabilities of the 'bad' rows which are not the best response and increase the probabilities of 'good' (best response) rows: Let

$$\begin{aligned} c_i(\mathbf{x}, \mathbf{y}) &= \max\left(0, \mathbf{A}_{i,*}\mathbf{y} - \mathbf{x}^T\mathbf{A}\mathbf{y}\right) \\ x_i' &= \frac{x_i + c_i(\mathbf{x}, \mathbf{y})}{1 + \sum_{j \in [m]} c_j(\mathbf{x}, \mathbf{y})} \end{aligned}$$

Note that c_i is positive when playing row i is better than $\mathbf{x}$.

Similarly for player B:

$$\begin{aligned} d_i(\mathbf{x}, \mathbf{y}) &= \max\left(0, \mathbf{x}^T\mathbf{B}_{*,i} - \mathbf{x}^T\mathbf{B}\mathbf{y}\right) \\ y_i' &= \frac{y_i + d_i(\mathbf{x}, \mathbf{y})}{1 + \sum_{j \in [n]} d_j(\mathbf{x}, \mathbf{y})} \end{aligned}$$

This is clearly a continuous map since the numerator is a linear function, and the denominator is quadratic. Thus the map is continuous. □

Next we go back to prove the second property.

Claim 4. $(\mathbf{x}, \mathbf{y})$ *is a Nash equilibrium of* G *if it is a fixed point of* f*, meaning that*

$$\begin{aligned} x_i &= \frac{x_i + c_i(\mathbf{x}, \mathbf{y})}{1 + \sum_{j \in [m]} c_j(\mathbf{x}, \mathbf{y})}, \quad \forall i \in [m] \qquad \textit{and} \\ y_i &= \frac{y_i + d_i(\mathbf{x}, \mathbf{y})}{1 + \sum_{j \in [n]} d_j(\mathbf{x}, \mathbf{y})}, \quad \forall i \in [n] \end{aligned}$$

Proof. Assume $\mathbf{x}$ is not a best response. Given the vector $[\mathbf{A}_{1,*}\mathbf{y} \ldots \mathbf{A}_{m,*}\mathbf{y}]$, we know that one entry, let's say the first entry, must be larger than $\mathbf{x}^T\mathbf{A}\mathbf{y}$. This implies that $c_1(\mathbf{x}, \mathbf{y}) > 0$. Furthermore, another entry of the vector, let's say the last entry, must be less than or equal to $\mathbf{x}^T\mathbf{A}\mathbf{y}$. This implies that $c_m(\mathbf{x}, \mathbf{y}) = 0$, which is a contradiction:

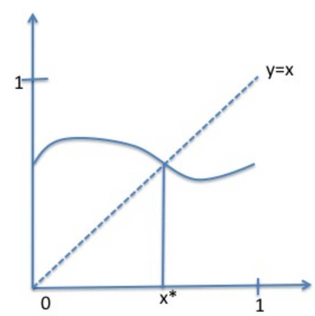
$$x_m > \frac{x_m + c_m(\mathbf{x}, \mathbf{y})}{1 + \sum_j c_j(\mathbf{x}, \mathbf{y})}$$

Thus, we must have a mixed equilibrium for this two-player game. □

2 Brouwer's Fixed Point Theorem [1911]

We begin with a simple 1-dimensional space, where $S = [0, 1]$ is the domain of the continous function $f : S \to S$.

Proof. We want to show that $\exists x^* \text{s.t.} f(x^*) = x^*$. If $f(0) = 0$, we know that our fixed point is 0. Otherwise,



$f(0) > 0$. If $f(1) = 1$, we are finished with the proof because 1 is our fixed proof. Otherwise, $f(1) < 1$. Then, by the mean-value theorem, x^* above in the picture, that is, the intersection of f with $x = y$, must be a fixed point. □

2.1 Sperner's Lemma in 1-dimension

The proof for the 2-dimensional space is relatively more complex then the one for the 1-dimensional space. The original proof given by Brouwer was relatively long. However in 1928, Sperner came up with a better approach for this proof based on a parity argument. Instead of counting the number of objects and proving that the number of objects is positive, looking simply at the parity and proving that it is odd suffices. We will now give an alternative proof for the 1-dimensional case using Sperner's Lemma:

Lemma 5. *Given n points on a line, a coloring is proper if the color of the first point is red and the color of n is green. If a coloring is proper, there exists an interval with different colors.*

Proof. Given an interval $i = [i, i + 1]$, we let

$$\text{RedGreen}(i) = \begin{cases} 1 & \text{if } [i, i+1] \text{ is red-green} \\ 0 & \text{otherwise} \end{cases}$$

and let

$$\text{RedPoints}(i) = \text{number of red points in } [i, i + 1].$$

Then we will show that

$$\sum_{i=1}^{n-1} \text{RedGreen}(i) \overset{\text{parity}}{\equiv} \sum_{i=1}^{n-1} \text{RedPoints}(i) = \text{odd}$$

Observe that RedPoints(i) must be $\in \{0, 1, 2\}$. We will now claim that RedGreen(i) has the same parity as RedPoints(i) at any segment $i = [i, i+1]$:

Case 1: RedGreen(i) $= 1 \Rightarrow$ RedPoints(i) $= 1$
Case 2: RedGreen(i) $= 0 \Rightarrow$ RedPoints(i) $\in \{0, 2\}$

Since the parity is the same in the above cases, the sum of all the segments must have the same parity. Now, we just need to show that:

$$\text{parity of } \sum_{i=1}^{n-1} \text{RedPoints}(i) = \text{odd}$$

To do this, we look at a point i on the interval that is not an endpoint. If the point is red, we count once for segment $[i, i-1]$, and again for segment, $[i, i+1]$. Thus, the total contribution of point i to the sum above is even, and the parity of the sum of segments is not affected.

Finally, we look at the two endpoints. We use the boundary condition (from the definition of a proper coloring) to find that the sum must be odd. □

2.2 Sperner's Lemma in 2-dimensions

We will now extend this simple argument to 2-dimensions. We start with a large triangle that we divide into progressively smaller triangles. We then have $\Theta(n^2)$ small triangles for every n intervals. We say a 3-coloring is proper if the thee vertices of the big triangle are colored in different colors (red, green, blue). Additionally, the red-green side of the triangle must have only red or green points, the red-blue side of the triangle must have only red or blue points, and the green-blue side must have only green or blue points. We can now state Sperner's Lemma:

Lemma 6. *If the 3-coloring is proper, $\exists$ a triangle which is a trichromatic triangle (an inner triangle vertices are colored with all three colors).*

To prove this, we will count the number of triangles, prove that number is odd, then it follows that you can always find at least one trichromatic triangle.

Proof. For each small triangle $\triangle$, we let

$$\text{Trichromatic}(\triangle) = \begin{cases} 1 & \text{if } \triangle \text{ is trichromatic} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\text{RedGreenSeg}(\triangle) = \text{number of red-green sides of } \triangle \in \{0, 1, 2, 3\}.$$

First we show that

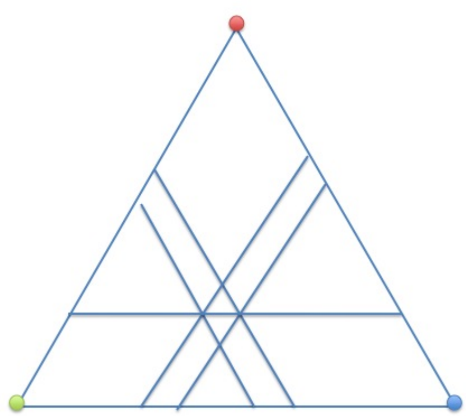
$$\sum_{all\triangle} \text{Trichromatic}(\triangle) \overset{\text{same parity}}{\sim} \sum_{all\triangle} \text{RedGreenSeg}(\triangle)$$

For any given $\triangle$, we have the following two cases:

case 1: $\text{Trichromatic}(\triangle) = 1 \Rightarrow \text{RedGreenSeg}(\triangle) = 1$

case 2: $\text{Trichromatic}(\triangle) = 0 \Rightarrow \text{RedGreenSeg}(\triangle) \in \{0, 2\}$

Now, we need to show that the parity of $\text{RedGreenSeg}(\triangle)$ is odd.
Any segment inside the triangle is shared by two small triangles and so is counted twice. Thus, either 0



or 2 is added to the sum, and the parity is unaffected.

As a result the relevant segments are the ones on the boundary of the outer triangle. RedGreenSegs can only occur on the red-green side of the triangle, so the red-blue sides and the green-blue sides are not useful either.

The red-green segments can only occur on the red-green side of the triangle. As these can only contribute to a single triangle, and since the number of RedGreenSegs on the red-green side is odd (using the 1-dimensional Sperner's lemma we just proved) we get the following, which concludes the proof:

$$\sum_{all\triangle} \text{RedGreenSig}(\triangle) = \text{odd}$$

$\square$